

# Reduction of incompressible Navier-Stokes system to viscous Burger's system

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## Abstract

This paper reduces the incompressible Navier-Stokes system to viscous Burger's system. The reduction is obtained via choosing a suitable pressure and requiring a certain type antisymmetry for the Navier-stokes system. Paper also briefly describes the effect of the chosen pressure on the respective Euler equations.

## 1 Navier-Stokes equations

The Navier-Stokes equations represent a simplified model of fluid dynamics in physics. The equation system is nonlinear and very hard to

solve, for additional information, see [1] . The incompressible Navier-Stokes is in vector form

$$\frac{\partial \vec{u}}{\partial t} = \nu \Delta \vec{u} - \vec{u} \cdot \nabla \vec{u} - \nabla p + \vec{f} \quad (1)$$

$$\nabla \cdot \vec{u} = 0 \quad (2)$$

where  $\vec{u} = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t))$  is the velocity field  $\vec{u} : \mathbb{R}^3 \times [0, \infty) \mapsto \mathbb{R}^3$  and  $p : \mathbb{R}^3 \times [0, \infty) \mapsto \mathbb{R}$  is the scalar pressure.  $\nu > 0$  is the viscosity which is constant. The vector  $\vec{f}$  is the external force field which plays no key role in this paper. We assume that the velocity field is smooth at least for some finite time.

## **2 Suitable pressure choice and symmetry properties of the Jacobian matrix**

Assume that the pressure is of the following functional form:

$$p(x, y, z, t) = \frac{1}{2}(\vec{u} \cdot \vec{u}) \quad (3)$$

The gradient field of pressure can be therefore represented as

$$\frac{\partial p}{\partial x} = uu_x + vv_x + ww_x \quad (4)$$

$$\frac{\partial p}{\partial y} = uu_y + vv_y + ww_y \quad (5)$$

$$\frac{\partial p}{\partial z} = uu_z + vv_z + ww_z \quad (6)$$

This means that the gradient vector can be represented as

$$\nabla p = \begin{pmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = J^T \vec{u} \quad (7)$$

where  $J$  is the familiar Jacobian matrix and  $J^T$  denotes its transpose.

On the other hand we have the convection vector

$$\vec{u} \cdot \nabla \vec{u} = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = J \vec{u} \quad (8)$$

So that the Navier-Stokes system can be written in compact form:

$$\frac{\partial \vec{u}}{\partial t} = \nu \Delta \vec{u} - J \vec{u} - J^T \vec{u} + \vec{f} = \nu \Delta \vec{u} - (J + J^T) \vec{u} + \vec{f} \quad (9)$$

### 3 Some consequences of the chosen pressure

Consider now the matrix

$$S = J + J^T = \begin{pmatrix} u_x + u_x & u_y + v_x & u_z + w_x \\ v_x + u_y & v_y + v_y & v_z + w_y \\ w_x + u_z & w_y + v_z & w_z + w_z \end{pmatrix} \quad (10)$$

We note immediately that matrix  $S$  is symmetric and also from the incompressibility condition it follows that the trace of the matrix is zero. According to the finite-dimensional spectral theorem, the eigenvectors of  $S$  are orthogonal and the eigenvalues are real [3]. Moreover, for matrices with zero trace the following holds

$$\sum_i \lambda_i = 0 \quad (11)$$

where  $\lambda_i \in \mathbb{R}$

$\forall i = 1, 2, 3$  denotes the  $i$ th eigenvalue of  $S$ . This means that there exists at least one eigenvalue  $\lambda_i < 0$ , unless all eigenvalues are zero.

### 3.1 The matrix $S$ and the Euler equations

Consider now the following representation:

$$S\vec{e} = \lambda\vec{e} \quad (12)$$

where  $\vec{e}$  is an eigenvector corresponding with some negative eigenvalue  $\lambda$ .

Take now the inviscid Navier-Stokes equations with no external force field, that is  $\nu = 0$  and  $\vec{f} = 0$ .

$$\frac{\partial \vec{u}}{\partial t} = -(J + J^T)\vec{u} \quad (13)$$

This is the so-called Euler's equation in fluid dynamics. Now take  $\vec{u} = \vec{e}$ .

That is, we consider the evolution of the eigenvector:

$$\frac{\partial \vec{e}}{\partial t} = -(J + J^T)\vec{e} = -S\vec{e} \quad (14)$$

Because of the eigenvalue equation, one can restate the equation:

$$\frac{\partial \vec{e}}{\partial t} = -\lambda \vec{e} \quad (15)$$

Imagine now that  $\vec{e}$  is the initial data for the Euler system. In other words,  $\vec{e}$  does not depend on  $t$ . If one then integrates the Euler equation up to time  $T$ , one sees that initial data will be scaled up by the factor

$$e^{-\int_0^T \lambda dt} \quad (16)$$

as  $\lambda$  is always less than zero, the initial velocity field will be scaled up. In particular, suppose that the initial velocity field satisfies the bounded energy condition

$$\int_{\mathbb{R}^3} |\vec{e}|^2 dx < C \quad (17)$$

for some  $C > 0$  at  $t = 0$ . Then it is obvious that the kinetic energy of the system increases without bound, as  $T \rightarrow \infty$ . In other words, the evolution for a particular initial velocity field determined by an eigenvector of the matrix  $S$ , blows up in terms of kinetic energy.

## 4 Reduction to viscous Burger's system

Let us now return to reduction of N-S to the viscous Burger's system.

We now want the matrix  $S$  to be diagonal, this holds, iff

$$u_y = -v_x \quad (18)$$

$$u_z = -w_x \quad (19)$$

$$v_x = -u_y \quad (20)$$

$$v_z = -w_y \quad (21)$$

$$w_x = -u_z \quad (22)$$

$$w_y = -v_z \quad (23)$$

Consider now the Viscosity vector  $\Delta \vec{u}$ . Using the diagonality conditions, one notices the following equivalence:

$$\begin{pmatrix} u_{xx} + u_{yy} + u_{zz} \\ v_{xx} + v_{yy} + v_{zz} \\ w_{xx} + w_{yy} + w_{zz} \end{pmatrix} = \begin{pmatrix} u_{xx} - v_{xy} - w_{xz} \\ -u_{yx} + v_{yy} - w_{yz} \\ -u_{zx} - v_{zy} + w_{zz} \end{pmatrix} \quad (24)$$

As we assume that the vector field  $\vec{u}$  is smooth, one can interchange the order of partial differentiation and take out the spatial differential

operator for each row:

$$\begin{pmatrix} u_{xx} + u_{yy} + u_{zz} \\ v_{xx} + v_{yy} + v_{zz} \\ w_{xx} + w_{yy} + w_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} u_x - \frac{\partial}{\partial x} v_y - \frac{\partial}{\partial x} w_z \\ -\frac{\partial}{\partial y} u_x + \frac{\partial}{\partial y} v_y - \frac{\partial}{\partial y} w_z \\ -\frac{\partial}{\partial z} u_x - \frac{\partial}{\partial x} v_y + \frac{\partial}{\partial x} w_z \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} (u_x - v_y - w_z) \\ \frac{\partial}{\partial y} (-u_x + v_y - w_z) \\ \frac{\partial}{\partial z} (-u_x - v_y + w_z) \end{pmatrix} \quad (25)$$

By using the assumption that the divergence of the velocity field is zero (incompressibility condition)

$$u_x + v_y + w_z = 0 \quad (26)$$

one can substitute and finally get

$$\begin{pmatrix} u_{xx} + u_{yy} + u_{zz} \\ v_{xx} + v_{yy} + v_{zz} \\ w_{xx} + w_{yy} + w_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} (2u_x) \\ \frac{\partial}{\partial y} (2v_y) \\ \frac{\partial}{\partial z} (2w_z) \end{pmatrix} \quad (27)$$

This does it. The full Navier-Stokes system is now reduced to the following system

$$u_t + 2uu_x = 2\nu u_{xx} \quad (28)$$

$$v_t + 2vv_y = 2\nu v_{yy} \quad (29)$$

$$w_t + 2ww_z = 2\nu w_{zz} \quad (30)$$

We have taken  $\vec{f} = 0$  for convenience. The equations hold as long as  $J + J^T$  is a diagonal matrix and  $\nabla \cdot \vec{u} = 0$ .

## 5 Conclusions and further research

The full Navier-Stokes reduces to viscous Burger's system under suitable symmetry properties of the velocity field. It is now interesting to proceed to the viscous limit where  $\nu \rightarrow 0$  as then one has the corresponding Euler system of fluid dynamics reduced to the inviscous Burger's system. One can then proceed to study the possible shock wave properties of the reduced system as the inviscous Burger's equation is known relatively well [2].

## References

- [1] "Lifschitz, L D Landau Fluid Mechanics, Butterworth-Heinemann, ISBN- 10:0750627670"
- [2] "L. Evans, Partial Differential Equations, American Mathematical Society"
- [3] Kreyszig, Introductory Functional Analysis with Applications, Wiley 1989